

**$(p, d)$ -ELLIPTIC CURVES OF GENUS TWO**

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ABSTRACT. We study stable curves of arithmetic genus 2 which admit two morphisms of finite degree  $p$ , resp.  $d$ , onto smooth elliptic curves, with particular attention to the case  $p$  prime.

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## 1. INTRODUCTION

In this paper we consider stable curves of arithmetic genus 2 which admit a  $(p, d)$ -elliptic configurations, namely two morphisms of finite degree  $p$ , resp.  $d$ , onto smooth elliptic curves  $D$  and  $E$ :

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow g \\ E & \xleftarrow[p:1]{} & \xrightarrow[d:1]{} D \end{array} .$$

A curve of genus 2 is called  $(p, d)$ -elliptic curve if it admits a  $(p, d)$ -elliptic configuration. If  $p = 2, d = 3$  we use the terminology bi-tri-elliptic curve.

Frey and Kani in [FK91] studied genus 2 coverings of elliptic curves, i.e., genus 2  $d$ -elliptic curves. A fundamental tool in their research was the observation that given a degree  $d$  map  $f: C \rightarrow E$  onto an elliptic curve then there exists a complementary map  $f': C \rightarrow E'$  of the same degree  $d$  onto a second elliptic curve (see §2). Later Kani in [Kan97] studied in details the arithmetic properties of such  $(d, d)$ -elliptic configurations, also providing existence results.

In this paper we study  $(p, d)$ -elliptic configurations following the construction described by Frey and Kani in [FK91].

It will turn out that stable  $(p, d)$ -elliptic curves of arithmetic genus 2 are automatically of compact type, i.e., they have compact Jacobian; thus in the first section we recall the Frey-Kani construction, noting that it extends to curves of compact type.

In Section 3 we study  $(p, d)$ -elliptic curves, with particular attention to the case  $p$  prime. In Theorem 3.12 we give a classification of such curves and in Section 3.C

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we show that for every pair of integers  $p, d > 1$  there exists a smooth  $(p, d)$ -elliptic curve of genus 2.

The original motivation for this article was the study of bi-tri-elliptic configurations, which parametrise certain strata in the boundary of the moduli space of stable Godeaux surfaces (see [FPR16]). Thus we describe the geometry of bi-tri-elliptic configurations in a little more detail in the last section.

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## 2. $d$ -ELLIPTIC CURVES OF GENUS 2

Here we recall and slightly refine some results from [FK91, §1], where the focus is on smooth curves and on the case  $d$  odd (see below).

**2.A. Set-up and preliminaries.** We work over an algebraically closed field  $\mathbb{K}$  whose characteristic does not divide the degree  $d$  of the finite morphisms that we consider. Throughout all this section  $C$  is a stable curve of genus 2 and  $J = J(C)$  is the Jacobian of  $C$ .

**Definition 2.1** — Let  $d \geq 2$  be an integer. We say that  $C$  is *d-elliptic* if there exists a finite degree  $d$  morphism  $f: C \rightarrow E$  such that  $E$  is a smooth curve of genus 1 and  $f$  does not factor through an étale cover of  $E$ ; we call  $f$  a *d-elliptic map*. Sometimes, a *d-elliptic map* is called an “elliptic subcover” and the curve  $C$  is said to have an “elliptic differential” (cf. [Kan97]); our choice of terminology is due to the fact that we wish to emphasize the degree  $d$  of the map. For  $d = 2, 3$ , the curve  $C$  is also called *bi-elliptic*, resp. *tri-elliptic*.

An isomorphism of  $d$ -elliptic curves  $f_i: C_i \rightarrow E_i$ ,  $i = 1, 2$ , is a pair of isomorphisms  $\varphi: C_1 \rightarrow C_2$  and  $\bar{\varphi}: E_1 \rightarrow E_2$  such that  $f_2 \circ \varphi = \bar{\varphi} \circ f_1$ .

For an abelian variety  $A$  we denote by  $A[d]$  its subgroup of  $d$ -torsion points. If  $A$  is principally polarised then there is a non degenerate alternating pairing  $e_d: A[d] \times A[d] \rightarrow \mu_d$  (where  $\mu_d$  denotes the  $d$ -th roots of unity) called Weil pairing (or Riemann form [Mum74, Ch.IV, §20]).

If  $A'$  is an abelian variety, then we call a group homomorphism  $\alpha: A[d] \rightarrow A'[d]$  *anti-symplectic* if for every  $P, Q \in A[d]$  one has:

$$e_d(\alpha(P), \alpha(Q)) = e_d(P, Q)^{-1},$$

or, equivalently, if the graph of  $\alpha$  is an isotropic subgroup of  $(A \times A')[d]$ .

**2.B. The Frey-Kani construction.** Now assume that  $C$  is a stable genus 2 curve of compact type, i.e., it is either smooth or the union of two elliptic curves intersecting in one point. Notice that the Jacobian  $J = J(C)$  is a principally polarised abelian surface.

Let  $f: C \rightarrow E$  be a  $d$ -elliptic map on  $C$ . The pull back map  $f^*: E \rightarrow J$  is injective, hence the norm map  $f_*: J \rightarrow E$  has connected kernel  $E'$ . Since the composition  $f_* f^*: E \rightarrow E$  is multiplication by  $d$ , the abelian subvarieties  $E'$  and  $f^* E$  of  $J$  intersect in  $E[d]$  and we have a tower of isogenies

$$(2.2) \quad E \times E' \xrightarrow[d^2:1]{h} J \xrightarrow[d^2:1]{h'} E \times E' \quad ,$$

whose composition is multiplication by  $d$ . Composing the Abel-Jacobi map  $C \hookrightarrow J$  with the projection to  $E'$  we get a second  $d$ -elliptic map  $f': C \rightarrow E'$ , which we call the *complementary  $d$ -elliptic map*.

The construction that follows, which we call the Frey-Kani construction, has been described in [FK91, §1] for smooth curves, but the proof works verbatim for stable curves of compact type. Therefore one has:

**Proposition 2.3** — *Let  $C$  be a stable  $d$ -elliptic curve of genus 2 of compact type, let  $f: C \rightarrow E$  and  $f': C \rightarrow E'$  be complementary  $d$ -elliptic maps and let  $h: E \times E' \rightarrow J = J(C)$  be as in (2.2). Then:*

- (i) *there exists an anti-symplectic isomorphism  $\alpha: E[d] \rightarrow E'[d]$  such that  $\ker h$  is the graph  $H_\alpha$  of  $\alpha$ ;*
- (ii) *the principal polarization on  $J$  pulls back to  $d(E \times \{0\} + \{0\} \times E')$ .*

Notice that if  $d = 2$ , then any isomorphism  $\alpha$  as in Proposition 2.3 is anti-symplectic. More generally, for a prime  $d$  the number of anti-symplectic isomorphisms  $E[d] \rightarrow E'[d]$  is equal to  $d(d^2 - 1)$  (cf. [FK91]).

The above proposition has a converse (see [FK91]):

**Proposition 2.4** — *Let  $E, E'$  be elliptic curves and let  $\alpha: E[d] \rightarrow E'[d]$  be an anti-symplectic isomorphism. Denote by  $H_\alpha$  the graph of  $\alpha$ ; set  $A := (E \times E')/H_\alpha$  and denote by  $h: E \times E' \rightarrow A$  the quotient map.*

*Then*

- (i)  *$d(E \times \{0\} + \{0\} \times E')$  descends to a principal polarization  $\Theta$  on  $A$ ;*
- (ii) *let  $C$  be a Theta-divisor on  $A$ ; then  $C$  is a stable curve of genus 2 of compact type and the maps  $f: C \rightarrow E$  and  $f': C \rightarrow E'$  induced by the natural maps  $A \rightarrow E$  and  $A \rightarrow E'$  are complementary  $d$ -elliptic maps.*
- (iii) *if  $d$  is odd, then there is precisely one symmetric Theta-divisor on  $A$  that is linearly equivalent to  $d(E \times \{0\} + \{0\} \times E')$*

**2.C. Special geometry for small  $d$ .** It is an interesting question under what conditions the polarisation coming from the Frey-Kani construction is reducible.

We answer this question for  $d = 2$ .

**Lemma 2.5** — *Let  $A$  be constructed as in Proposition 2.4 for  $d = 2$ . Then the principal polarization  $\Theta$  of  $A$  is reducible if and only if there exists an isomorphism  $\psi: E' \rightarrow E$  such that the map  $E \times E' \rightarrow E \times E$  defined by  $(x, y) \mapsto (x, \psi(y))$  maps  $H_\alpha$  to the subgroup  $\Delta[2] = \{(\eta, \eta) | \eta \in E[2]\}$ .*

*Moreover, up to isomorphism the bi-elliptic map  $f$  is given by the composition*

$$C = E \times \{0\} \cup \{0\} \times E \hookrightarrow J(C) = E \times E \xrightarrow{+} E,$$

*that is, it is the identity on each component of  $C$ ; the complementary map  $f'$  is the identity on one component and multiplication by  $-1$  on the other.*

*Proof.* Denote by  $\bar{E}, \bar{E}'$  the image of  $E, E'$  in  $A$ : we have  $\Theta = (\bar{E} + \bar{E}')/2$  and  $\bar{E}\bar{E}' = 4$ . Assume that  $\Theta = B_1 + B_2$  is reducible, with  $B_1, B_2$  elliptic curves such that  $B_1B_2 = 1$  (recall that in this case  $A \cong B_1 \times B_2$  as ppav's). Then  $1 = B_1\Theta = B_1(\bar{E} + \bar{E}')/2$  implies either  $B_1\bar{E} = B_1\bar{E}' = 1$  or, say,  $B_1\bar{E} = 2, B_1\bar{E}' = 0$ . The latter case cannot occur, since we would have  $B_1 = \bar{E}'$ , hence  $2 = B_1\bar{E} = \bar{E}'\bar{E} = 4$ , a contradiction.

Hence composing the map  $E \rightarrow \bar{E}$  with the projections  $A \rightarrow B_i$ , for  $i = 1, 2$  one obtains isomorphisms  $\varphi_i: E \rightarrow B_i$ . Hence we have an isomorphism  $A \rightarrow E \times E$  that maps  $\bar{E}$  to the diagonal in  $E \times E$  and  $\bar{E}'$  to the graph of an automorphism

$\sigma$  of  $E$ . Since  $\bar{E}\bar{E}' = 4$ ,  $\sigma$  has 4 fixed points. By the classification of the possible automorphism groups of an elliptic curve, it follows that  $\sigma$  is multiplication by  $-1$  and  $\bar{E}'$  is mapped to the antidiagonal. Composing  $E' \rightarrow A$  with the isomorphisms  $A \rightarrow E \times E$  and with the first projection  $E \times E \rightarrow E$  we obtain the required isomorphism  $\psi: E' \rightarrow E$ . Indeed it is not hard to check that the following diagram commutes:

$$(2.6) \quad \begin{array}{ccc} E \times E' & \xrightarrow{(\text{id}, \psi)} & E \times E \\ \downarrow h & & \downarrow q \\ A & \longrightarrow & E \times E \end{array},$$

where  $h$  is the quotient map and  $q(x, y) = (x + y, x - y)$ .

Conversely, assume that  $E = E'$  and  $\alpha$  is the identity. The map  $q: E \times E \rightarrow E \times E$  defined by  $q(x, y) = (x + y, x - y)$  has kernel  $H_\alpha = \Delta[2]$ , hence  $A$  is isomorphic to  $E \times E$ . Let  $C = E \times \{0\} + \{0\} \times E$ ; then  $q^*C = \Delta + \Delta^-$ , where  $\Delta$  is the diagonal and  $\Delta^-$  is the antidiagonal. Since  $(q^*C)^2 = 8$  by the pull-back formula and  $q^*C(E \times \{0\} + \{0\} \times E) = 4$ , the divisors  $q^*C$  and  $2(E \times \{0\} + \{0\} \times E)$  are algebraically equivalent by the Index Theorem, hence  $C$  is the principal polarisation of Proposition 2.4. (More precisely, since  $q^*C$  and  $2(E \times \{0\} + \{0\} \times E)$  restrict to the same divisor on  $E \times \{0\}$  and  $\{0\} \times E$  they are actually linearly equivalent). The final part of the statement follows.  $\square$

We close this section with an alternative description of bi-elliptic curves of genus 2 of compact type, which basically stems from the fact that a double cover is the quotient by an involution.

**Lemma 2.7** — *Let  $C$  be a genus 2 stable curve of compact type, let  $f: C \rightarrow E$  and  $f': C \rightarrow E'$  be complementary bi-elliptic maps and let  $\sigma$ , resp.  $\sigma'$ , be the involution induced by  $f$ , resp.  $f'$ . Then the group  $\langle \sigma, \sigma' \rangle$  is isomorphic to  $\mathbb{Z}/2 \times \mathbb{Z}/2$  and  $\tau := \sigma\sigma'$  is the hyperelliptic involution.*

*Proof.* Let  $J$  be the Jacobian of  $C$ . The involution  $\sigma$  on  $J$  is induced by the involution  $(x, y) \mapsto (x, -y)$  of  $E \times E'$  (cf. (2.2)) and, similarly,  $\sigma'$  is induced by  $(x, y) \mapsto (-x, y)$ . So  $\tau = \sigma\sigma'$  acts as multiplication by  $-1$  on  $J$  and therefore, if  $C$  is smooth, is the hyperelliptic involution. If  $C$  is reducible, then  $\tau$  is multiplication by  $-1$  on both components of  $C$ .

Since  $\tau$  is in the center of  $\text{Aut}(C)$ ,  $\sigma$  and  $\sigma'$  commute and  $\langle \sigma, \sigma' \rangle$  has order 4.  $\square$

It is well known that bidouble covers can be reconstructed from their building data (cf. [Par91, §2]); we explain this in the case at hand.

Choose points  $P_1, P_2$  and distinct points  $Q_1, Q_2, Q_3 \in \mathbb{P}^1$  that are also distinct from  $P_1$  and  $P_2$  ( $P_1$  and  $P_2$  are allowed to coincide). Let  $\pi: C \rightarrow \mathbb{P}^1$  be the bidouble cover branched on  $D_1 = P_2$ ,  $D_2 = P_1$  and  $D_3 = Q_1 + Q_2 + Q_3$ , denote by  $G$  the Galois group of  $\pi$  and by  $\sigma \in G$  (resp.  $\sigma'$  and  $\tau$ ) the involution the fixes the preimage of  $D_1$  (resp.  $D_2, D_3$ ). Assume first that  $P_1 \neq P_2$ ; in this case  $C$  is a smooth curve of genus 2 and for  $i = 1, 2$  the quotients  $E = C/\sigma$  and  $E' = C/\sigma'$  are smooth curves of genus 1. The involution  $\tau$  has 6 fixed points and therefore is the hyperelliptic involution.

If  $P_1 = P_2$ , then  $C$  has a node over  $P_1 = P_2$  and the normalization is the bidouble cover of  $\mathbb{P}^1$  with branch divisors  $D_1 = D_2 = 0$  and  $D_3 = P_1 + Q_1 + Q_2 + Q_3$ . So

$C$  is reducible and has two components, both isomorphic to the double cover of  $\mathbb{P}^1$  branched on  $P_1 + Q_1 + Q_2 + Q_3$ .

This construction is related to the construction given in Proposition 2.4 as follows.

Let  $\pi: C \rightarrow \mathbb{P}^1$  be as above, with  $C$  smooth, and take the preimage of  $P_1$  as the origin  $0 \in E$  and the preimage of  $P_2$  as the origin  $0' \in E'$ . Denote by  $A_1, A_2, A_3$  (resp. by  $B_1, B_2, B_3$ ) the preimages of  $Q_1, Q_2, Q_3$  in  $E$  (resp. in  $E'$ ). Then the nonzero elements of  $E[2]$  (resp.  $E'[2]$ ) are  $\eta_i := A_i - 0$  (resp.  $\eta'_i := B_i - 0'$ ),  $i = 1, 2, 3$ ; we define  $\alpha: E[2] \rightarrow E'[2]$  as the isomorphism that maps  $\eta_i \rightarrow \eta'_i$ .

We claim that the bi-elliptic structure on  $C$  is obtained via the Frey-Kani construction with the above choice of  $\alpha$ , i.e., the kernel of the pull-back map  $h^*: E \times E' \rightarrow J := J(C)$  is the graph  $H_\alpha$  of  $\alpha$ .

Indeed, since the kernel  $\Gamma$  of the pull-back map has order 4, it is enough to show that  $H_\alpha$  is contained in  $\Gamma$ . In addition one has  $f^*A_i = f'^*B_i$  for  $i = 1, 2, 3$ , hence we only need to show that  $f^*0$  and  $f'^*0'$  are linearly equivalent. The divisor  $f^*0$  is the ramification divisor of  $f$ , hence  $f^*0 \equiv K_C$ ; the same argument shows that  $f'^*0' \equiv K_C$  and we are done.

The case  $C$  reducible is obvious.

*Remark 2.8* — For tri-elliptic curves one can apply the general theory of triple covers [Mir85] to deduce the following result [FPR16, Lem. 2.8]: a stable curve  $C$  of genus 2 admits a tri-elliptic map  $C \rightarrow E$  such that  $C$  embeds into the symmetric square of  $E$  as a tri-section of the Albanese map  $S^2E \rightarrow E$ .

Note however that a tri-elliptic map  $C \rightarrow E$  cannot be a cyclic cover, since by the Hurwitz formula it would be ramified over precisely one point and this is impossible, for instance, by [Par91, Prop. 2.1]). So there is no elementary description of  $C$  just in terms of the ramification divisor.

### 3. (p, d)-ELLIPTIC CURVES OF GENUS 2

We consider stable curves of genus 2 admitting two distinct maps to elliptic curves.

#### 3.A. (n, d)-elliptic curves and configurations.

**Definition 3.1** — Let  $C$  be a stable curve of genus 2. An  $(n, d)$ -elliptic configuration  $(C, f, g)$  is a diagram

$$(3.2) \quad \begin{array}{ccc} & C & \\ f \swarrow & & \searrow g \\ E & \xleftarrow{n:1} \quad \xrightarrow{d:1} & D \end{array},$$

where  $f$  is an  $n$ -elliptic map and  $g$  is a  $d$ -elliptic map such that there is no isomorphism  $\psi: E \rightarrow D$  such that  $g = \psi \circ f$ . We refer to  $C$  as to an  $(n, d)$ -elliptic curve (of genus 2).

An isomorphism of  $(n, d)$ -elliptic configurations is an isomorphism of diagrams like (3.2).

**Lemma 3.3** — *If  $C$  is  $(n, d)$ -elliptic stable curve of genus 2, then it is of compact type.*

*Proof.* As usual, we let  $f: C \rightarrow E$  and  $g: C \rightarrow D$  be the  $n$ -elliptic and  $d$ -elliptic maps.

The curve  $C$  cannot have rational components since it has a finite map onto an elliptic curve, hence it is enough to rule out the possibility that  $C$  is irreducible with one node. Assume by contradiction that this is the case, let  $C^\nu \rightarrow C$  be the normalization map and denote by  $\psi_1: C^\nu \rightarrow E$  (resp.  $\psi_2: C^\nu \rightarrow D$ ) the map of degree  $n$  (resp.  $d$ ) induced by  $f$  (resp.  $g$ ). We denote by  $O, P \in C^\nu$  the points of that map to the node of  $C$  and we take  $O$  as the origin; with a suitable choice of the origins in  $E$  and  $D$  we can assume that  $\psi_1$  and  $\psi_2$  are isogenies.

Since  $\psi_1$  and  $\psi_2$  factor through  $C^\nu \rightarrow C$ ,  $P$  belongs to  $\ker \psi_1 \cap \ker \psi_2$  and for  $i = 1, 2$   $\psi_i$  factors through the étale covers  $C^\nu / \langle P \rangle \rightarrow E$  and  $C^\nu / \langle P \rangle \rightarrow D$ . It follows that  $f$  and  $g$  also factor through  $C^\nu / \langle P \rangle \rightarrow E$  and  $C^\nu / \langle P \rangle \rightarrow D$  hence, by the definition of  $d$ -elliptic curve, it follows that  $n = d = 2$  and the two bi-elliptic structures differ by an isomorphism  $E \rightarrow D$ , a contradiction.  $\square$

*Remark 3.4* — By the Frey-Kani construction given in §2, if  $C$  is of compact type and has a  $d$ -elliptic map  $f: C \rightarrow E$  then, if we denote  $f': C \rightarrow E'$  complementary  $d$ -elliptic map,  $(C, f, f')$  is a  $(d, d)$ -elliptic configuration. We refer to this as to the *trivial  $(d, d)$ -elliptic configuration*.

**3.B.  $(p, d)$ -elliptic curves.** Assume we are given an  $(n, d)$ -elliptic configuration as in (3.2), which is non-trivial in the sense of Remark 3.4. Then both elliptic maps factor, up to isomorphism, through the Abel-Jacobi map of  $C$  and are thus uniquely determined by the subgroups  $\ker f_*$  and  $\ker g_*$ .

To analyse such a configuration, we apply the Frey-Kani construction to the  $n$ -elliptic map given by  $f$  and then we consider  $\bar{F} = \ker g_*$ . We can extend (2.2) to a diagram

$$(3.5) \quad \begin{array}{ccccccc} F & \xrightarrow{\quad h_F \quad} & \bar{F} & & & & \\ \downarrow (\varphi, \varphi') & & \downarrow & \searrow m:1 & & & \\ E \times E' & \xrightarrow{\quad h \quad} & J(C) & \xrightarrow{\quad f_* \times f'_* \quad} & E \times E' & \longrightarrow & E \\ & & \downarrow g_* & & & & \\ & & D & & & & \end{array}$$

where  $\bar{F}\Theta = d$  ( $\Theta$  denoting the principal polarization) and  $F$  is the connected component of  $h^{-1}F$  containing the origin. Indeed we have the following

*Remark 3.6* — A genus 2 curve of compact type has an  $n$ -elliptic structure if and only if its Jacobian  $J$  contains a connected 1-dimensional subgroup  $\bar{E}'$  such that  $\bar{E}'\Theta = n$ . Therefore an  $n$ -elliptic curve  $C$  has an  $(n, d)$ -elliptic structure if and only if  $J$  contains a second connected 1-dimensional subgroup  $\bar{F}$  such that  $\bar{F}\Theta = d$  and  $\bar{F} \neq \bar{E}'$ . So, if we exclude the trivial  $(d, d)$ -structures (cf. Remark 3.4), the Jacobian of an  $(n, d)$ -elliptic curve of genus 2 contains at least three, hence infinitely many, connected 1-dimensional subgroups. In particular the curve  $C$  has infinitely many elliptic structures, and the curves  $E$  and  $E'$  are isogenous.

For a given  $(n, d)$ -elliptic configuration  $(C, f, g)$  we denote  $\bar{F} = \ker g_*$  and  $\bar{E}' = \ker f_*$  and we define the *twisting number* of  $(C, f, g)$  as

$$(3.7) \quad m = m(C, f, g) := \bar{F} \ker f_* = \bar{F} \bar{E}' = \deg(\bar{F} \rightarrow E) = n^2 \frac{\deg \varphi}{\deg h_F}$$

where  $\varphi$  and  $h_F$  are as in (3.5).

*Remark 3.8* — One has  $m > 0$ , by the definition of  $(n, d)$ -elliptic curve.

Denote by  $\bar{E}$  the kernel of  $f'_*: J \rightarrow E'$ , where  $f'$  is the complementary map of  $f$ . Then by the Frey-Kani construction we have  $\Theta = \frac{\bar{E} + \bar{E}'}{n}$ , hence  $nd = n\bar{F}\Theta = m + \bar{F}\bar{E}$ . It follows that  $m \leq nd$ , with equality holding if and only if  $\bar{F}\bar{E} = 0$ , namely iff we are in the trivial case  $g = f'$ .

We first provide three examples with  $m < nd$  that fit into this general pattern and then prove that when  $n = p$  is a prime these cover all possibilities for non trivial elliptic configurations.

**Example 3.9** — Let  $n, d$  be integers. Let  $F$  be an elliptic curve and let  $\varphi: F \rightarrow E$  and  $\varphi': F \rightarrow E'$  be isogenies such that:

- $\ker \varphi \cap \ker \varphi' = \{0\}$ , hence  $\varphi \times \varphi': F \rightarrow E \times E'$  is injective;
- $\deg \varphi + \deg \varphi' = nd$ , and  $n$  and  $\deg \varphi$  are coprime.

We abuse notation and denote again by  $F$  the image of  $\varphi \times \varphi'$ . The subgroup  $H := F[n] \subset (E \times E')[n]$  satisfies  $H \cap E = H \cap E' = \{0\}$ , since  $EF = \deg \varphi'$  and  $E'F = \deg \varphi$  are coprime to  $n$ . Hence  $H$  is the graph of an isomorphism  $E[n] \rightarrow E'[n]$ . The polarization  $n(E \times \{0\} + \{0\} \times E')$  restricts on  $F$  to a divisor of degree  $n^2d$ , which therefore is a pull back via the map  $F \rightarrow F$  defined by multiplication by  $n$ . By the functorial properties of the Weil pairing (see statement (1) of [Mum74, Ch.IV, §23, p.228]) it follows that  $F[n]$  is an isotropic subspace of  $(E \times E')[n]$ .

Let  $A = (E \times E')/H$  and let  $\Theta$  be the principal polarization of  $A$  (see Proposition 2.4). Denote by  $\bar{F}$  the image of  $F$  in  $A$ : then we have  $n^2\bar{F}\Theta = nF(E \times \{0\} + \{0\} \times E') = n^2d$ , namely  $\bar{F}\Theta = d$ . By Remark 3.6 we obtain an  $(n, d)$ -elliptic configuration with twisting number  $m = n^2 \frac{\deg \varphi}{\deg h_F} = \deg \varphi$  (cf. (3.7)).

From now on we will focus on the case where  $n = p$  is a prime number.

**Example 3.10** — Let  $p, d$  be integers and assume that  $p$  is a prime. Let  $F$  be an elliptic curve and let  $\varphi: F \rightarrow E$  and  $\varphi': F \rightarrow E'$  be isogenies such that:

- $\ker \varphi \cap \ker \varphi' = \{0\}$ ;
- $\deg \varphi + \deg \varphi' = d$ ;
- $F[p] \not\subset \ker \varphi$  and  $F[p] \not\subset \ker \varphi'$ .

Under the above conditions, it is possible to find an antisymplectic isomorphism  $\alpha: E[p] \rightarrow E'[p]$  such that  $H_\alpha \cap F$  has order  $p$ , where  $H_\alpha$  is the graph of  $\alpha$ . This follows because by our assumptions there exists  $0 \neq v \in F[p]$  such that  $v \notin \ker \varphi \cup \ker \varphi'$ . Moreover, since the Weil pairing of a product is given by the product of the Weil pairings (see statement (2) of [Mum74, Ch.IV, §23, p.228]), the annihilator  $W$  of  $v$  in  $(E \times E')[p]$  does not contain  $E[p] \times \{0\}$  nor  $\{0\} \times E'[p]$ . The linear subspace  $W$  is three dimensional, hence  $\mathbb{P}(W)$  is a projective plane over  $\mathbb{F}_p$ . Now consider in  $\mathbb{P}(W)$  the pencil  $\mathcal{F}$  of lines through  $[v]$ : since  $\mathcal{F}$  consists of  $p+1$  lines, if  $p > 2$  there is at least a line  $l \in \mathcal{F}$  that does not intersect the lines  $r := \mathbb{P}(E[p] \times \{0\})$  and  $s := \mathbb{P}(\{0\} \times E'[p])$  and distinct from  $t := \mathbb{P}(F[p])$ . The subspace of  $(E \times E')[p]$  corresponding to  $l$  is the graph of an isomorphism  $\alpha: E[p] \rightarrow E'[p]$  and is also isotropic, hence  $\alpha$  is anti-symplectic. For  $p = 2$ , any isomorphism  $\alpha$  is antisymplectic, hence it is enough to find a line in  $\mathbb{P}((E \times E')[2])$  that contains  $[v]$ , which is distinct from  $t$  and does not intersect  $r$  and  $s$ . An elementary geometrical argument shows that there exist two lines with this property.

Therefore we can consider  $A := (E \times E')/H_\alpha$  and the principal polarization  $\Theta$  of  $A$  (cf. Proposition 2.4). Again we denote by  $\bar{F}$  the image of  $F$  in  $A$ , obtaining



$p^2 \bar{F}\Theta = pF(p(E \times \{0\} + \{0\} \times E')) = p^2 d$ , namely  $\bar{F}\Theta = d$ , i.e. by Remark 3.6 we get a  $(p, d)$ -elliptic configuration. In this case by (3.7) we have  $m = p^2 \frac{\deg \varphi}{\deg h_F} = p \deg \varphi$ .

**Example 3.11** — Let  $p, d$  be integers such that  $p$  is a prime and  $d$  is divisible by  $p$ . Write  $d = p\delta$  and let  $F$  be an elliptic curve with  $\varphi: F \rightarrow E$  and  $\varphi': F \rightarrow E'$  isogenies such that:

- $\ker \varphi \cap \ker \varphi' = \{0\}$ ;
- $\deg \varphi + \deg \varphi' = \delta$ ;

We look for an antisymplectic isomorphism  $\alpha: E[p] \rightarrow E'[p]$  such that  $H_\alpha \cap F = \{0\}$ ,  $H_\alpha$  being the graph of  $\alpha$ .

To see that such  $\alpha$  exists we argue as follows. As in Example 3.10 we identify 2-dimensional subspaces of  $(E \times E')[p]$  with lines in  $\mathbb{P}^3(\mathbb{F}_p) := \mathbb{P}((E \times E')[p])$ . We have seen in Example 3.10 that there are  $p + 1$  isotropic lines through any point, hence there exist  $(p + 1)(p^2 + 1)$  isotropic lines.

The isotropic lines meeting a given line are  $p(p + 1) + 1$  or  $(p + 1)^2$ , according to whether the line is isotropic or not. Set  $r := \mathbb{P}(E[p] \times \{0\})$ ,  $s := \mathbb{P}(\{0\} \times E'[p])$  and  $t = \mathbb{P}(F[p])$ ; note that  $r$  and  $s$  are not isotropic. Hence there are at most  $3(p + 1)^2$  isotropic lines meeting  $r \cup s \cup t$ . However, all the lines joining a point of  $r$  and a point of  $s$  are isotropic hence, by subtracting these lines (that we had counted twice) we get the better upper estimate  $2(p + 1)^2$  for the number of isotropic lines meeting  $r \cup s \cup t$ . For  $p \geq 3$  this shows the existence of the isotropic subspace  $H_\alpha$  that we are looking for, since  $(p + 1)(p^2 + 1) - 2(p + 1)^2 = (p + 1)(p^2 - 2p - 1) > 0$ .

For  $p = 2$ , we need only find a line that is disjoint from  $r \cup t \cup s$ . We observe that  $\mathbb{P}^3(\mathbb{F}_2)$  contains 35 lines, that the lines intersecting a given line are 19, that the lines meeting two given skew lines are 9 and the lines meeting three mutually skew lines are 3.

If  $\deg \varphi$  and  $\deg \varphi'$  are odd, then the three lines  $r$ ,  $s$  and  $t$  are mutually skew: then the set of lines meeting at least one of these consists of  $3 \cdot 19 - 3 \cdot 9 + 3 = 33$  lines, hence there are 2 possibilities for  $H_\alpha$ .

Now assume that both  $\deg \varphi$  and  $\deg \varphi'$  are even. In this case  $t$  meets both  $r$  and  $s$ . The number of lines intersecting  $r \cup t$  is equal to  $7 + 7 - 3 = 11$ , since there are 7 lines in plane spanned by  $r$  and  $t$ , there are 7 lines passing through  $r \cap t$ , and 3 lines common to these two sets. An analogous argument shows that there are  $3 + 3 - 1 = 5$  lines meeting  $t$ ,  $r$  and  $s$ . So the number of lines intersecting  $r \cup t \cup s$  is equal to  $3 \cdot 19 - 9 - 2 \cdot 11 + 5 = 31$ , so there are 4 possibilities for  $H_\alpha$ .

Finally we consider the case where  $\deg \varphi$  is even and  $\deg \varphi'$  is odd. There are two possibilities: either  $r = t$  or  $r$  and  $t$  are coplanar but distinct. In the former case, the number of lines meeting  $r \cup t \cup s = r \cup s$  is equal to  $2 \cdot 19 - 9 = 29$ , so there are 6 possibilities for  $H_\alpha$ . In the latter case, the number of lines meeting  $r \cup t \cup s$  is equal to  $3 \cdot 19 - 2 \cdot 9 - 11 + 5 = 33$ , so there are 2 possibilities for  $H_\alpha$ .

Taking  $A := (E \times E')/H_\alpha$  and  $\Theta$  the principal polarization of  $A$  (cf. Proposition 2.4) and denoting by  $\bar{F}$  the image of  $F$  in  $A$ , we get  $p^2 \bar{F}\Theta = p^2 F(p(E \times \{0\} + \{0\} \times E')) = p^2 d$ , namely  $\bar{F}\Theta = d$ , i.e. by Remark 3.6 we get a  $(p, d)$ -elliptic configuration. In this case (3.7) yields  $m = p^2 \frac{\deg \varphi}{\deg h_F} = p^2 \deg \varphi$ .

**Theorem 3.12** — Let  $p$  be a prime and let  $d$  be a positive integer. Let  $C$  be a stable curve of genus 2 and let  $(C, f, g)$  be a non-trivial (cf. Remark 3.4)  $(p, d)$ -elliptic



configuration

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow g \\ E & & D \end{array}, \quad \begin{array}{c} p:1 \\ d:1 \end{array}$$

Denote by  $\bar{E}'$  (resp.  $\bar{F}$ ) the kernel of  $f_*: J = J(C) \rightarrow E$  (resp.  $g_*: J \rightarrow D$ ) and let  $m = \bar{E}'\bar{F}$  be the twisting number as in (3.7). Then

- (i) the  $(p, d)$ -elliptic configuration arises as in Example 3.9, or 3.10, or 3.11, and thus  $1 \leq m \leq pd - 1$ ;
- (ii) the case of Example 3.11 can occur only if  $p$  divides  $d$  and  $p^2$  divides  $m$ ;
- (iii) the case of Example 3.9 occurs if and only if  $m$  is not divisible by  $p$ .

*Proof.* By Remark 3.8 we have  $0 < m \leq pd$ , and  $m = pd$  holds only in the trivial case  $g = f'$ . Therefore by our assumptions it is  $1 \leq m \leq pd - 1$ .

We use freely the notation of §3.B and diagram 3.5 and we denote by  $\varphi: F \rightarrow E$  and  $\varphi': F \rightarrow E'$  the isogenies induced by the two projections of  $E \times E'$ . Note that  $\ker \varphi \cap \ker \varphi' = \{0\}$  by construction. The pull-back  $h^*\bar{F} \subset E \times E'$  is algebraically equivalent to  $\nu F$  for some integer  $\nu \in \{1, p, p^2\}$  (one has  $p^2 = \nu|H \cap F|$ ). We have  $p^2m = p^2\bar{F}\bar{E}' = \nu F(p^2(\{0\} \times E'))$ , i.e.,  $m = \nu F(\{0\} \times E') = \nu \deg \varphi$ . In the same way, one obtains  $pd - m = \nu F(E \times \{0\}) = \nu \deg \varphi'$ . In particular,  $\nu = 1$  if  $m$  is not divisible by  $p$ .

Consider the case  $\nu = 1$ , i.e.  $H = F[p]$ . In this case the map  $E \times E' \rightarrow J(C)$  induces a degree  $p^2$  isogeny  $F \rightarrow \bar{F} \cong F$ , the degree of  $\varphi$  is equal to  $m$  and the degree of  $\varphi'$  is equal to  $pd - m$ . Since  $H$ , being a graph, intersects  $E \times \{0\}$  and  $\{0\} \times E'$  only in 0, it follows that  $m$ , which is equal to the order of  $(\{0\} \times E') \cap F$ , is prime to  $p$ , and the same is true for  $\deg \varphi' = pd - m$ . So,  $C$  is constructed as in Example 3.9.

Next, assume that  $\nu = p$ , i.e.  $H \cap F$  has order  $p$ . In this case, one has  $m = p \deg \varphi$  and  $\deg \varphi + \deg \varphi' = d$ . Since  $H \cap F$  has order  $p$  and  $H$  is a graph, it follows that  $F[p] \not\subset \ker \varphi$  and  $F[p] \not\subset \ker \varphi'$ , hence  $C$  is constructed as in Example 3.10.

Finally consider the case  $\nu = p^2$ . In this case one has  $m = p^2 \deg \varphi$  and  $d = p(\deg \varphi + \deg \varphi')$ , hence  $C$  is constructed as in Example 3.11.  $\square$

**3.C. Existence of smooth  $(n, d)$ -elliptic curves.** First of all let us recall that an irreducible  $(n, d)$ -elliptic curve is smooth by Lemma 3.3.

By Lemma 2.5, for  $n = 2$  a necessary condition for the irreducibility of the genus 2 curve  $C$  constructed as in Proposition 2.4 is that the curves  $E$  and  $E'$  are isomorphic, hence if  $E$  does not have complex multiplication then the constructions of Examples 3.9, 3.10 and 3.11 yield examples of smooth  $(2, d)$ -elliptic curves of genus 2 for every  $d > 2$ .

In general, it is not clear whether the constructions of Examples 3.9, 3.10 and 3.11 give rise to irreducible, hence smooth, curves. We are able to settle this point at least in a special case:

**Proposition 3.13** — *Let  $n \geq 2, d \geq 3$  be integers; let  $E$  be an elliptic curve without complex multiplication,  $\xi \in E$  an element of order  $r := dn - 1$ , and  $\varphi': E \rightarrow E' := E / \langle \xi \rangle$  the quotient map.*

*Then the  $(n, d)$ -elliptic genus 2 curve constructed as in Example 3.9 with  $F = E$ ,  $\varphi = \text{Id}_E$  and  $\varphi'$  as above is smooth.*

As an immediate consequence we obtain:

**Corollary 3.14** — *For every pair of integers  $n, d > 1$  there exists a smooth  $(n, d)$ -elliptic curve of genus 2 with twisting number  $m = 1$ .*

*Proof.* For  $n = d = 2$  the claim follows by Lemma 2.5, for instance by using the construction of Example 3.9, and by Proposition 3.13 in the remaining cases.  $\square$

*Proof of Proposition 3.13.* Denote by  $\Xi$  the product polarization on  $E \times E'$ . Set  $H := \{(\eta, \varphi'(\eta)) \mid \eta \in E[p]\}$  and let  $h: E \times E' \rightarrow A := (E \times E')/H$  be the quotient map.

We argue by contradiction, so assume that the principal polarization of  $A$  induced by  $n\Xi$  is reducible and denote it by  $C = C_1 + C_2$ . Up to a translation we may assume that the singular point of  $C$  is the origin of  $A$ . Let  $\tilde{C}_i$  be the connected component of the preimage of  $C_i$  containing the origin of  $E \times E'$ ,  $i = 1, 2$ , so that  $h^*C_i$  is numerically equivalent to  $\nu_i \tilde{C}_i$  for a positive integer  $\nu_i$ . One has

$$(3.15) \quad n^2 = \nu_i |H \cap \tilde{C}_i| \quad \text{and} \quad \frac{n}{\nu_i} = \tilde{C}_i \Xi \in \mathbb{Z}.$$

Since  $E$  does not have complex multiplication ( $\text{End } E = \mathbb{Z}$ ), the connected 1-dimensional subgroups of  $E \times E$  distinct from  $E \times \{0\}$  and  $\{0\} \times E$  are of the form  $\{(ax, bx) \mid x \in E\}$ , with  $a, b$  coprime integers. This is well known, but we give a quick proof for lack of a suitable reference. Let  $G$  be such a subgroup, and denote by  $\psi_i: G \rightarrow E$ ,  $i = 1, 2$  the isogenies induced by the two projections. Note that  $\ker \psi_1 \cap \ker \psi_2 = \{0\}$ . If  $G$  is isomorphic to  $E$ , then the  $\psi_i$  are multiplication maps and  $G$  is of the form  $D_{a,b}$  for some pair of coprime integers  $a, b$ . So assume that  $G$  and  $E$  are not isomorphic and consider an isogeny  $\chi: E \rightarrow G$ . Since  $\chi$  is not a multiplication map, there exists an integer  $k$  and elements  $u, v \in E[k]$  such that  $\chi(u) = 0$  and  $\chi(v) = v' \neq 0$ . Now consider the maps  $\mu_i := \psi_i \circ \chi: E \rightarrow E$ , which are multiplication maps by integer  $t_i$ ,  $i = 1, 2$ . Both  $t_1$  and  $t_2$  are divisible by  $k$ , since for  $i = 1, 2$  we have  $\mu_i(u) = 0$ , hence  $\psi_i(v') = \mu_i(v) = 0$  and so  $v' = 0$ , a contradiction.

Since  $D_{a,b} = D_{-a,-b}$ , we may always assume  $a \geq 0$ . It follows that the connected 1-dimensional subgroups of  $E \times E'$  distinct from  $E \times \{0\}$  and  $\{0\} \times E'$  are of the form  $D_{a,b} = \{(ax, b\varphi'(x)) \mid x \in E\}$ . Notice that the kernel of the induced map  $E \rightarrow D_{a,b}$  is the cyclic subgroup of  $\langle \xi \rangle$  of order  $\delta := g.c.d.(a, r)$ . Using this observation one computes:

$$(3.16) \quad D_{a,b}(\{0\} \times E') = \frac{a^2}{\delta}, \quad D_{a,b}(E \times \{0\}) = \frac{b^2 r}{\delta}, \quad D_{a,b} D_{1,1} = (b-a)^2 \frac{r}{\delta}.$$

For  $i = 1, 2$ , let  $a_i, b_i \in \mathbb{Z}$  be such that  $\tilde{C}_i = D_{a_i, b_i}$ , with  $a_i \geq 0$ ; set  $\delta_i = g.c.d.(a_i, r)$ . We will now derive a contradiction using intersection numbers.

**Step 1:** We have  $a_i > 0$ . Indeed if  $a_i = 0$  we have  $\tilde{C}_i \Xi = 1$  and  $|H \cap \tilde{C}_i| = 1$ , so (3.15) gives  $n = \nu_i$  and  $\nu_i = n^2$ , against our assumptions.

**Step 2:** We show  $(a_i, b_i) \neq (1, 1)$ . Indeed, assuming  $\tilde{C}_i = D_{1,1}$  (3.15) gives

$$\frac{n}{\nu_i} = \tilde{C}_i \Xi = D_{1,1} \Xi = 1 + r = dn,$$

which is impossible since  $d > 1$ . In particular, since  $a_i$  and  $b_i$  are coprime, we have  $a_i \neq b_i$ .

**Step 3:** From the above steps we derive two inequalities and a divisibility property which will lead to a contradiction.

First of all we have, for  $i = 1, 2$ ,

$$(3.17) \quad n | \nu_i (a_i - b_i),$$

Indeed, since  $D_{1,1} \cap \tilde{C}_i$  is a subgroup containing  $H \cap \tilde{C}_i$  we have that  $\tilde{C}_i D_{1,1} = (b_i - a_i)^2 \frac{r}{\delta_i}$  is divisible by  $\frac{n^2}{\nu_i}$ , hence  $(b_i - a_i)^2$  is divisible by  $\frac{n^2}{\nu_i}$ , since  $\frac{r}{\delta_i}$  is an integer prime to  $n$ . So  $\nu_i^2 (b_i - a_i)^2$  is divisible by  $n^2$ , and therefore  $\nu_i(a_i - b_i)$  is divisible by  $n$ .

Secondly, by (3.16) we have  $n = (\nu_1 \tilde{C}_1 + \nu_2 \tilde{C}_2)(\{0\} \times E') = \nu_1 a_1 \frac{a_1}{\delta_1} + \nu_2 a_2 \frac{a_2}{\delta_2}$  and  $n = (\nu_1 \tilde{C}_1 + \nu_2 \tilde{C}_2)(E \times \{0\}) = \nu_1 b_1^2 \frac{r}{\delta_1} + \nu_2 b_2^2 \frac{r}{\delta_2}$ . In particular, we have

$$(3.18) \quad \nu_1 a_1 + \nu_2 a_2 \leq n, \quad d(\nu_1 b_1^2 + \nu_2 b_2^2) \leq n,$$

since  $\frac{r}{\delta_i}$  is an integer and  $\frac{r}{\delta_i} \geq d \frac{n}{a_i} - 1 > d - 1$ .

**Step 4:** We cannot have  $b_i > 0$ . Indeed in this case, since  $0 \leq \nu_i a_i, \nu_i b_i < n$  by (3.18) and  $n$  divides the difference  $\nu_i a_i - \nu_i b_i$  by (3.17), then we necessarily have  $a_i = b_i$  contradicting Step 2.

**Step 5:** We cannot have  $b_i \leq 0$ . Indeed the same argument as in the previous step shows that we would necessarily have  $\nu_i b_i = \nu_i a_i - n$  for  $i = 1, 2$ . By (3.18) we may assume, say,  $\nu_1 a_1 \leq \frac{n}{2}$  and thus by the above equality  $\nu_1 |b_1| = -\nu_1 b_1 \geq \frac{n}{2}$ . Then (3.18) gives:

$$n \geq d \nu_1 b_1^2 \geq |b_1| \frac{dn}{2},$$

a contradiction since  $d > 2$ .

Combining the last two steps we arrive at a contradiction and have thus proved that that the polarisation is irreducible and hence is a smooth  $(n, d)$ -elliptic curve of genus 2.  $\square$

#### 4. BI-TRI-ELLIPTIC CURVES

For the applications to the classification of Gorenstein stable Godeaux surfaces the case of bi-tri-elliptic configurations is of particular interest. In this section we first formulate Theorem 3.12 in this case and then analyse reducible bi-tri-elliptic curves in more detail.

Indeed we have the following characterization of reducible bi-tri-elliptic configurations.

**Corollary 4.1** — *Let  $(C, f, g)$  be a bi-tri-elliptic configuration on a stable curve of arithmetic genus 2. Then the twisting number  $m$  defined in (3.7) satisfies  $1 \leq m \leq 5$  and there are the following possibilities:*

- (a)  $m$  is odd and the configuration arises as in Example 3.9 with  $\deg \varphi = m$ ;
- (b)  $m = 2\mu$  is even and the configuration arises as in Example 3.10 with  $\deg \varphi = \mu$ .

*Remark 4.2* — Counting parameters we see that the space of bi-tri-elliptic configurations is one-dimensional, but we did not consider its finer structure, e.g., the number of irreducible or connected components.

Now assume that  $C \cong E \cup_0 E$ , where  $E$  is an elliptic curve with a degree 2 endomorphism  $\psi: E \rightarrow E$ . Then we can build a natural bi-tri-elliptic configuration

$$(4.3) \quad \begin{array}{ccc} & E \cup_0 E & \\ f = \text{id} \cup \text{id} \swarrow & & \searrow g = \text{id} \cup \psi \\ E & \xleftarrow{2:1} & \xrightarrow{3:1} E \end{array}$$

We will now show that every bi-tri-elliptic configuration  $(C, f, g)$  with  $C$  reducible is of this form. Indeed, by Lemma 2.5 the bi-elliptic map  $f$  on the reducible curve  $C$  is isomorphic to the composition of horizontal arrows in the diagram

$$\begin{array}{ccccc} & & \bar{F} & & \\ & & \downarrow & & \\ C = E \times \{0\} \cup \{0\} \times E & \hookrightarrow & E \times E & \xrightarrow{+} & E \\ & \searrow g & \downarrow & & \\ & & D & & \end{array} ,$$

and the tri-elliptic map is uniquely determined by the subgroup  $\bar{F}$ . Note that the covering involution of  $f$  exchanges the components of  $C$ .

We have  $\bar{F}C = 3$  and without changing  $f$  we can assume that  $\bar{F}(\{0\} \times E) = 1$  and  $\bar{F}(E \times \{0\}) = 2$ . In other words,  $\bar{F}$  is the graph of a degree 2 endomorphism  $\psi: E \rightarrow E$  and  $E \times \{0\}$  is identified with the second elliptic curve  $D$  by the restriction of  $g$ . Therefore the bi-tri-elliptic configurations is as in (4.3).

An isomorphism from  $(C, f, g)$  to another bi-tri-elliptic configuration  $(\tilde{C} = \tilde{E} \cup_0 \tilde{E}, \tilde{f}, \tilde{g})$  such that  $\tilde{f}$  is the identity on each component of  $\tilde{C}$  is uniquely determined by an isomorphism  $E \cong \tilde{E}$  and thus we have proved the first part of the following

**Proposition 4.4** — *The above construction induces a bijection on the set of isomorphism-classes of bi-tri-elliptic configurations  $(C, f, g)$  with  $C$  a reducible stable curve of genus 2 and the set  $\{(E, \psi)\}$  of elliptic curves together with an endomorphism of degree 2.*

For every  $1 \leq m \leq 5$  there are exactly two such pairs  $(E, \psi)$ , which are listed in Table 1, thus in total there are 10 isomorphism classes of bi-tri-elliptic configurations with  $C$  a reducible stable curve of genus 2.

TABLE 1. Endomorphisms of degree 2 on elliptic curves

$E = \mathbb{C}/\Gamma$	$\Gamma = \text{End}(E)$	$\xi$	$m$
$E_1$	$\mathbb{Z}[i]$	$-1 \pm i$	1
		$1 \pm i$	5
$E_2$	$\mathbb{Z}[i\sqrt{2}]$	$\pm i\sqrt{2}$	3
$E_3$	$\mathbb{Z}[\frac{1}{2}(1 + i\sqrt{7})]$	$-\frac{1}{2}(1 \pm i\sqrt{7})$	2
		$\frac{1}{2}(1 \pm i\sqrt{7})$	4

*Proof.* We need to recall some elementary facts about endomorphisms of elliptic curves. Details can be found for example in [Sil09, Ch.11] or [Sil94, Ch.II]. Any endomorphism  $\psi$  of an elliptic curve  $E$  is given by multiplication by a complex number  $\xi$  and this embeds  $\text{End } E \hookrightarrow \mathbb{C}$  as a maximal order in an imaginary quadratic number field  $K \cong \text{End}(E) \otimes \mathbb{Q}$ . The degree of the endomorphism  $\psi$  coincides with the norm  $N_{K/\mathbb{Q}}(\xi)$ .

Thus elements inducing an endomorphism of degree 2 are characterised as those  $\xi \in \mathbb{C} \setminus \mathbb{R}$  that are integral over  $\mathbb{Z}$  with characteristic polynomial

$$p_\xi(t) = t^2 - \text{trace}_{K/\mathbb{Q}}(\xi)t + N_{K/\mathbb{Q}}(\xi) = t^2 - 2\text{Re}(\xi)t + 2 \in \mathbb{Z}[t].$$

This gives exactly the elements listed in Table 1 and each one of them is contained in a unique maximal order by [Sil09, Example 11.3.1] (see also [Sil94, Prop. 2.3.1]).

It remains to compute the invariant  $m$ , which is in our case the intersection of  $\Gamma_\psi = \bar{F} \subset E \times E$  with the kernel of the addition map, that is, the anti-diagonal. Thus  $m$  equals the number of fixed points of the endomorphism  $-\psi$ , which by the holomorphic Lefschetz fixed-point formula [GH78, Ch. 3.4] gives

$$m = \sum_{i=0}^2 (-1)^i \text{trace}(-\psi_*|_{H_i(E, \mathbb{Q})}) = 1 - \text{trace}_{K/\mathbb{Q}}(-\xi) + N_{K/\mathbb{Q}}(-\xi) = p_\xi(-1),$$

because every fixed point of  $\psi$  is simple.  $\square$

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